

Error amplification in solving Prony system with near-colliding nodes

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ABSTRACT. We consider a reconstruction problem for “spike-train” signals F of the a priori known form $F(x) = \sum_{j=1}^d a_j \delta(x - x_j)$, from their moments $m_k(F) = \int x^k F(x) dx$. We assume $m_k(F)$ to be known for $k = 0, 1, \dots, 2d - 1$, with an absolute error not exceeding $\epsilon > 0$. This problem is essentially equivalent to solving the Prony system $\sum_{j=1}^d a_j x_j^k = m_k(F)$, $k = 0, 1, \dots, 2d - 1$.

We study the “geometry of error amplification” in reconstruction of F from $m_k(F)$, in situations where the nodes x_1, \dots, x_d near-collide, i.e. form a cluster of size $h \ll 1$. We show that in this case error amplification is governed by the “Prony leaves” $S_q F$, in the parameter space of signals F , along which the first $q + 1$ moments remain constant. On this base we produce accurate, up to the constants, lower and upper bounds on the worst case reconstruction error, and show how the Prony leaves can help in improving reconstruction accuracy.

1. Introduction

The problem of reconstruction of spike-trains, and of similar signals, from noisy moment measurements, and a closely related problem of robust solving the classical Prony system, is a classical problem in Mathematics and Engineering. It is of major practical importance, and, in case when the nodes nearly collide, it is well known to present major mathematical difficulties. It is closely related to a spike-train “super-resolution problem”, (see [1, 2, 4, 5, 13, 14, 15, 16, 17, 18, 19, 20, 22, 24, 25, 27, 28] as a small sample).

The aim of the present paper is to describe the patterns of amplification of the measurements error ϵ in the reconstruction process, caused by the geometric nature of the Prony system, independently of the specific method of its inversion. Thus we assume that *Prony inversion (when possible) is accurate, and the reconstruction error is caused only by the measurements noise*. Moreover, we will always assume below that *all the “algebraic-geometric” operations, with the known parameters, are performed accurately*. Specifically this concerns constructing certain algebraic curves and higher-dimensional varieties. Of course, such algorithmic constructions

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in Computational Algebraic Geometry may present well-known difficulties, but in the present paper we do not touch this topic.

Specifically, we study, following the line of [1, 3], the case where the nodes of our spike-train signal F form a cluster of size $h \ll 1$. We introduce the “Prony leaves” $S_q F$, in the parameter space of signals F , which are algebraic varieties, defined by the first q equations of the Prony system, and show that the geometry of these leaves governs the error amplification. As one specific application, we produce accurate, up to the constants, lower and upper bounds on the worst case error in reconstruction of F . Next, we show how the Prony leaves can help in improving reconstruction accuracy, when certain additional information is available. We study in some detail the case of two nodes, providing explicit description of the Prony curves, and of the behavior of the signals along them.

1.1. Setting of the problem. Assume that our signal $F(x)$ is a spike-train, i.e. a linear combination of d shifted δ -functions:

$$(1.1) \quad F(x) = \sum_{j=1}^d a_j \delta(x - x_j),$$

where $a = (a_1, \dots, a_d) \in \mathbb{R}^d$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. We assume that the form (1.1) is a priori known, but the specific parameters (a, x) are unknown. Our goal is to reconstruct (a, x) from $2d$ moments $m_k(F) = \int_{-\infty}^{\infty} x^k F(x) dx$, $k = 0, \dots, 2d - 1$, which are known with a possible error $\epsilon > 0$.

An immediate “symbolic computation” shows that the moments $m_k(F)$ are expressed through the unknown parameters (a, x) as $m_k(F) = \sum_{j=1}^d a_j x_j^k$. Hence our reconstruction problem is equivalent to solving the *Prony system* of algebraic equations, with the unknowns a_j, x_j :

$$(1.2) \quad \sum_{j=1}^d a_j x_j^k = m_k(F), \quad k = 0, 1, \dots, 2d - 1.$$

This system appears in many theoretical and applied problems. There exists a vast literature on Prony and similar systems - see, as a very small sample, [7]-[9], [28]-[32] and references therein.

We shall denote by $\mathcal{P} = \mathcal{P}_d$ the parameter space of signals F ,

$$\mathcal{P}_d = \{(a, x) = (a_1, \dots, a_d, x_1, \dots, x_d) \in \mathbb{R}^{2d}, \quad x_1 < x_2 < \dots < x_d\},$$

and by $\mathcal{M} = \mathcal{M}_{2d-1} \cong \mathbb{R}^{2d}$ the moment space, consisting of the $2d$ -tuples of the moments $(m_0, m_1, \dots, m_{2d-1})$. We will identify signals F with their parameters $(a, x) \in \mathcal{P}$.

1.2. Sketch of main results. Let a signal $F(x)$ as above be fixed. The main object we study in this paper is the ϵ -error set $E_\epsilon(F)$ consisting of all signals $F'(x)$ which may appear as the reconstruction of F , from the noisy moment measurements μ'_k with $|\mu'_k - m_k(F)| \leq \epsilon$, $k = 0, \dots, 2d - 1$.

DEFINITION 1.1. *The error set $E_\epsilon(F) \subset \mathcal{P}$ is the set, consisting of all the signals $F'(x) \in \mathcal{P}$ with*

$$|m_k(F') - m_k(F)| \leq \epsilon, \quad k = 0, \dots, 2d - 1.$$

In order to describe the geometry of the ϵ -error set $E_\epsilon(F)$ let us consider the moments $m_k = m_k(F')$, $k = 0, \dots, 2d - 1$ as non-linear coordinates in the space \mathcal{P} of signals F' . In Section 2 below, under certain additional assumptions on F , we justify this approach via the “quantitative inverse function theorem”.

Let us assume now that the nodes x_1, \dots, x_d of the signal F form a cluster of size $h \ll 1$, while inside the cluster the nodes are well separated from one another. We show that under this assumption the Prony reconstruction for the measurement error $\epsilon \leq O(h^{2d-1})$ can be accurately described in the moments coordinates m_k (inverse function theorem). (For $\epsilon \geq O(h^{2d-1})$ the Prony reconstruction becomes much more complicated. In particular, singularities of various types appear. We provide some results in this situation, in the case $d = 2$, in Section 4 below).

Now we define the “Prony leaves”, which are just the coordinate subspaces of different dimensions, with respect to the moment coordinates.

DEFINITION 1.2. *The Prony leave $S_q(F)$, $q = 0, \dots, 2d - 1$, is an algebraic subset of \mathcal{P} defined by the equations*

$$m_k(F') = m_k(F), \quad k = 0, \dots, q.$$

Equivalently, the Prony leaves are defined by the first q equations of the Prony system (1.2).

For a fixed F and decreasing q the Prony leaves $S_q(F)$ form an increasing chain of algebraic varieties in \mathcal{P} :

$$F \in S_{2d-1}(F) \subset S_{2d-2}(F) \subset \dots \subset S_1(\mu) \subset S_0(F) \subset \mathcal{P}.$$

Generically, $S_q(F)$ is a smooth subvariety of dimension $2d - q - 1$. In particular, $S_{2d-1}(F)$ is a finite collection of points (all the solutions of the Prony system (1.2)), while $S_{2d-2}(F)$ is a regular curve, passing through F .

Informally, our main results in the case $\epsilon \leq O(h^{2d-1})$ (Section 3) are the following:

1. *Let the nodes x_1, \dots, x_d of F form a cluster of size $h \ll 1$. Then for $\epsilon \leq O(h^{2d-1})$ the ϵ -error set $E_\epsilon(F)$ is a “non-linear coordinate parallelepiped” $\Pi_{h,\epsilon}(F)$ with respect to the moment coordinates $m_k(F')$, centered at F . Its width in the direction of the moment coordinate m_k , $k = 0, \dots, 2d - 1$, is of order ϵh^{-k} . In particular, the maximal stretching of $\Pi_{h,\epsilon}(F)$, of order $\epsilon h^{-(2d-1)}$, occurs along the Prony curve $S_{2d-2}(F)$.*

An immediate conclusion is that

2. *The worst case reconstruction error $\|F' - F\|$ is of order $\epsilon h^{-(2d-1)}$. However, the reconstructions F' with the error $\|F' - F\|$ of this order cannot occur everywhere: they fall into a small neighborhood (of a size $\epsilon h^{-(2d-3)}$) of the Prony curve $S_{2d-2}(F)$.*

Our next main result concerns the accuracy of reconstruction of the Prony leaves $S_q(F)$:

3. *The curve $S_{2d-2}(F)$ itself can be reconstructed with a better accuracy than the point reconstruction: that of order $\epsilon h^{-(2d-3)}$. The “hierarchy of the accuracy rates” is continued along the chain $S_{2d-1}(F) \subset \dots \subset S_0(F)$ of the Prony leaves $S_q(F)$: each $S_q(F)$ can be reconstructed with an accuracy of order ϵh^{-q} .*

We conclude our results in the scale $\epsilon \leq O(h^{2d-1})$ with an informal presentation of the following fact:

4. *If a certain additional a priori information is available on the signal F , the reconstruction accuracy can be significantly improved (in some cases) via the following procedure: first we reconstruct the Prony leave $S_q(F)$ for a certain appropriate q . The accuracy of this reconstruction (of order ϵh^{-q}) is higher than that for F . Then we use the additional information available in order to accurately localize the signal F inside the Prony leave $S_q(F)$.*

Now we consider the case $\epsilon \geq O(h^{2d-1})$ (Section 4). As it was mentioned above, our approach, based on a regularity of the moment coordinates, does not work any more, since for large errors the reconstruction encounters singularities.

Still, the moment coordinates, and, in particular, Prony leaves S_q continue to govern the error amplification in the Prony inversion, in much larger scales.

The moment coordinates, and Prony leaves, being algebraic objects, are defined globally. However, the inverse function theorem cannot help anymore in their description. Instead we have to explicitly describe these varieties, in particular, providing their explicit parametrization. Such a description is possible: in [3] we provide it in the case of $d = 2$, while in [6] we extend the explicit description of Prony leaves to any number of nodes.

However, in a strict contrast to the nonsingular part of the Prony foliation, which is, in a sense, *completely described* by the inverse function theorem, near singularities multiple complicated scenarios of error amplification are possible. In Section 4 some of these scenarios are described.

In particular,

5. *We explicitly describe two scenarios of the reconstruction error behavior along the Prony curves S_2 , for ϵ up to $\sim h^2$ (and not only up to $\sim h^3$, as in Section 3): escaping of one node to infinity, and nodes collision. We use this description to improve reconstruction accuracy if certain lower (upper) bounds on the amplitudes are a priori known. Some results of numerical simulations are presented.*

2. Prony mapping and its inversion

2.1. Prony mapping. We shall denote by $\mathcal{P} = \mathcal{P}_d$ the parameter space of signals F ,

$$\mathcal{P}_d = \{(a, x) = (a_1, \dots, a_d, x_1, \dots, x_d) \in \mathbb{R}^{2d}, x_1 < x_2 < \dots < x_d\},$$

and by $\mathcal{M} = \mathcal{M}_{2d-1} \cong \mathbb{R}^{2d}$ the moment space, consisting of the $2d$ -tuples of the moments $(m_0, m_1, \dots, m_{2d-1})$. We will identify signals F with their parameters $(a, x) \in \mathcal{P}$.

DEFINITION 2.1. *The Prony mapping $PM = PM_{2d-1} : \mathcal{P}_d \rightarrow \mathcal{M}_{2d-1}$ is given by*

$$PM(F) = \mu = (\mu_0, \dots, \mu_{2d-1}) \in \mathcal{M}, \mu_k = m_k(F), k = 0, \dots, 2d-1.$$

For $F \in \mathcal{P}$ the problem of its reconstruction from the exact moment measurements $\mu = (\mu_0, \dots, \mu_{2d-1}) \in \mathcal{M}$, is the problem of inverting the Prony mapping PM . In this paper we always assume that this inversion (when defined) is accurate.

Let us be given the noisy measurements $\mu' = (\mu'_0, \dots, \mu'_{2d-1}) \in \mathcal{M}$ of the moments of F . By our assumption, the measurement error of each of the moments

m_k does not exceed ϵ , i.e. $|\mu'_k - \mu_k(F)| \leq \epsilon$. Equivalently, the noisy measurement μ' may fall at any point of the cube

$$Q_\epsilon(\mu) = \{\mu' = (\mu'_0, \dots, \mu'_{2d-1}) \in \mathcal{M}, |\mu'_k - \mu_k| \leq \epsilon, k = 0, 1, \dots, 2d-1\}.$$

Consequently, the ϵ -error set $E_\epsilon(F)$ is the preimage $PM^{-1}(Q_\epsilon(\mu)) \subset \mathcal{P}$.

We will use the maximum norm $|\mu' - \mu| = \max_{k=0,1,\dots,2d-1} |\mu'_k - \mu_k|$ on the moment space \mathcal{M} . The Euclidean norm on the signal space \mathcal{P} will be denoted by $\|\cdot\|$.

2.2. Inverse Function theorem and its consequences. Our first result describes the inversion of the Prony mapping in a neighborhood of a “regular point”, i.e. of a signal G with all its d nodes well separated, and with all its amplitudes bounded and well separated from zero. This result is, essentially, a direct application of the “quantitative inverse function theorem” (see, for instance, [21], Theorem 3.2, and references therein), combined with the estimates of the Jacobian of the Prony mapping, given in [7].

Assume that the nodes x_1, \dots, x_d of G belong to the interval $I = [-1, 1]$, and for a certain η with $0 < \eta \leq \frac{1}{d+1}$, the distance between the neighbor nodes x_j, x_{j+1} , $j = 1, \dots, d-1$, is at least η . We also assume that for certain m, M with $0 < m < M$, the amplitudes a_1, \dots, a_d satisfy $m \leq |a_j| \leq M$, $j = 1, \dots, d$. We call such signals (η, m, M) -regular. We distinguish the parameter and the moment spaces of signals G , denoting them by $\bar{\mathcal{P}}, \bar{\mathcal{M}}$, respectively. For $G \in \bar{\mathcal{P}}$ we denote by $\nu = (\nu_1, \dots, \nu_{2d-1})$ its Prony image $PM(G) \in \mathcal{M}$.

THEOREM 2.1. *Let G be an (η, m, M) -regular signal. There exist constants R, C_1, C_2, C_3 , depending only on d, η, m, M such that*

1. *The Jacobian J at G of the Prony mapping PM is invertible, with*

$$\|J\|, \|J^{-1}\| \leq C_1.$$

2. *The inverse mapping PM^{-1} is regular analytic (and algebraic) in the cube $Q_R(\nu)$ of size R centered at $\nu \in \bar{\mathcal{M}}$, and satisfies there, for each $\nu', \nu'' \in Q_R(\nu)$*

$$C_2 |\nu'' - \nu'| \leq \|PM^{-1}(\nu'') - PM^{-1}(\nu')\| \leq C_3 |\nu'' - \nu'|,$$

Proof: The first statement follows from the estimates of the Jacobian J of the Prony mapping PM , provided in [7]. The second one follows from the “quantitative inverse function theorem” (see, for instance, [21]), taking into account that the required bounds on the first and the second derivatives of PM can be easily obtained in terms of d, η, m, M . \square

Let us denote $\Omega_R(G) \subset \bar{\mathcal{P}}$ the preimage $PM^{-1}Q_R(\nu)$. We give an equivalent formulation of Theorem 2.1, in terms of the moment coordinates.

DEFINITION 2.2. *For G a regular signal as above, and G' denoting signals near G , the moment coordinates are the functions $m_k = m_k(G')$. The moment metric $d(G', G'')$ on $\bar{\mathcal{P}}$ is defined through the moment coordinates as*

$$d(G', G'') := \max_{k=0}^{2d-1} |m_k(G'') - m_k(G')|.$$

Clearly, the moment metric $d(\cdot, \cdot)$ is induced, via the Prony mapping, from the norm $|\cdot|$ on $\bar{\mathcal{M}}$, i.e.

$$d(G', G'') = |PM(G'), PM(G'')|.$$

COROLLARY 2.1. *Let G be a regular signal as above. Then the moment coordinates $m_k = m_k(G')$ form a regular analytic coordinate system on $\Omega_R(G)$. The moment metric $d(G', G'')$ is Lipschitz equivalent on $\Omega_R(G)$ to the Euclidean metric $\|G'' - G'\|$:*

$$C_2 d(G', G'') \leq \|G'' - G'\| \leq C_3 d(G', G'').$$

3. The error set for nodes forming an h -cluster: main results

We use regular signals G as above, as a model for signals with a “regular cluster”: we say that a signal $F \in \mathcal{P}$ forms an (h, κ, η, m, M) -regular cluster, if it can be obtained from an (η, m, M) -regular signal G by an h -down-scaling, and then a shift to the point κ , with respect to the variable x .

It is convenient to formulate our main result in terms of the model signal G , i.e/ after a shift and rescaling of the nodes to $[-1, 1]$. Denote by $ME_\epsilon(G) \subset \bar{\mathcal{M}}$ the set of all the possible errors in the moments $m_k(G)$, $k = 0, 1, \dots, 2d-1$, corresponding to the errors not exceeding ϵ in the moments of F . Denote by $\bar{E}_\epsilon(G) \subset \bar{\mathcal{P}}$ the set consisting of all the signals G' for which $PM(G') \in ME_\epsilon(G)$. Clearly, $\bar{E}_\epsilon(G)$ is the image, under the κ -shift and $\frac{1}{h}$ -scaling, of the ϵ -error set $E_\epsilon(F) \subset \mathcal{P}$.

3.1. The case of a zero shift. Theorem 3.1 below describes the set $\bar{E}_\epsilon(G) \subset \bar{\mathcal{P}}$, under an additional assumption that there is no shift. In this case the description becomes especially transparent. The effect of a non-zero shift κ is described in Section 3.2 below. In particular, a version Theorem 3.1 with a non-zero shift is given in Theorem 3.2.

THEOREM 3.1. *Let $F \in \mathcal{P}$ form an $(h, 0, \eta, m, M)$ -regular cluster, and let R be the constant of Theorem 2.1. Then for each positive $\epsilon \leq Rh^{2d-1}$ the error set $\bar{E}_\epsilon(G) \subset \bar{\mathcal{P}}$ is a parallelepiped $\Pi_{\epsilon, h}(G)$ in moment coordinates in $\bar{\mathcal{P}}$, consisting of all G' satisfying the inequalities*

$$|m_k(G') - m_k(G)| \leq \epsilon h^{-k}, \quad k = 0, \dots, 2d-1.$$

For each $q = 0, \dots, 2d-1$ the intersection $\bar{E}_{\epsilon, q}(G)$ of the error set $\bar{E}_\epsilon(G)$ with the Prony leave $S_q(G)$ is given in $S_q(G)$ by the inequalities

$$|m_k(G') - m_k(G)| \leq \epsilon h^{-k}, \quad k = q+1, \dots, 2d-1.$$

The edges of the parallelepiped $\bar{E}_\epsilon(G)$, which are not in S_q , are not longer than ϵh^{-q} . In particular, $\bar{E}_\epsilon(G)$ is contained in the ϵh^{-q} -neighborhood, with respect to the moment distance d (or in the $C_2 \epsilon h^{-q}$ -neighborhood, with respect to the Euclidean distance) of the q -th error set $E_{\epsilon, q}(G)$.

Proof: First we describe the moment error set $ME_\epsilon(G) \subset \bar{\mathcal{M}}$. By definition, it consists of all the possible errors in the moments $m_k(G)$, $k = 0, 1, \dots, 2d-1$, corresponding to the errors not exceeding ϵ in the moments of F .

Consider the scaling transformation Sc_α , which acts on signals F via scaling of the argument: $Sc_\alpha(F)(x) = F(\alpha x)$. Immediate computation shows that for $F(x) = \sum_{j=1}^d a_j \delta(x - x_j)$ we have $Sc_\alpha(F) = \sum_{j=1}^d a_j \delta(x - \alpha x_j)$, and therefore

$$m_k(Sc_\alpha(F)) = \sum_{j=1}^d a_j \alpha^k x_j^k = \alpha^k m_k(F).$$

Accordingly, we define scaling transformation $Sc_\alpha^m : \mathcal{M} \rightarrow \mathcal{M}$ of the moment space as follows: for $\mu = (\mu_0, \dots, \mu_{2d-1})$

$$Sc_\alpha^m(\mu) = \nu = (\nu_0, \dots, \nu_{2d-1}), \quad \nu_k = \alpha^k \mu_k, \quad k = 0, \dots, 2d-1.$$

With these definitions the identity

$$PM(Sc_\alpha(F)) = Sc_\alpha^m(PM(F))$$

is satisfied.

For the model signal G we have $G = Sc_{1/h}(F)$. Accordingly, the set $ME_\epsilon(G)$ of the possible measurements for the moments of G is $Sc_{1/h}^m(ME_\epsilon(F))$. The initial moment error set $ME_\epsilon(F)$ is the ϵ -cube $Q_\epsilon(\mu) \in \mathcal{M}$, defined by

$$|m'_k - m_k| \leq \epsilon, \quad k = 0, 1, \dots, 2d-1.$$

Consequently, $ME_\epsilon(G)$ is a coordinate parallelepiped

$$M\Pi_{\epsilon,h}(\nu) := \{\nu' \in \mathcal{M}, |\nu'_k - \nu_k| \leq \epsilon h^{-k}, \quad k = 0, 1, \dots, 2d-1\}.$$

The error set $\bar{E}_\epsilon(G) \subset \bar{\mathcal{P}}$ is the preimage

$$\bar{E}_\epsilon(G) = PM^{-1}(ME_\epsilon(G)) = PM^{-1}(M\Pi_{\epsilon,h}(\nu)).$$

In order to apply Theorem 2.1 and Corollary 2.1 we have to check that the parallelepiped $M\Pi_{\epsilon,h}(\nu)$ is contained in the cube $Q_R(\nu)$ of size R centered at $\nu \in \bar{\mathcal{M}}$. But the maximal edge of $M\Pi_{\epsilon,h}(\nu)$ has length ϵh^{-2d+1} , and hence for $\epsilon \leq Rh^{2d-1}$ the required inclusion holds.

Now the first and the second statements of Theorem 3.1 follow from the description of $ME_\epsilon(G)$ given above, and from the fact that the moment coordinates, as well as the moment distance on $\bar{\mathcal{P}}$, are induced via the Prony mapping from the coordinates and the maximum norm on the moment space \mathcal{M} . The last statement, concerning the Euclidean metric on $\bar{\mathcal{P}}$, follows from the equivalence of the moment and Euclidean metrics, given in Corollary 2.1. \square

3.2. The case of a non-zero shift.

THEOREM 3.2. *Let $F \in \mathcal{P}$ form an (h, κ, η, m, M) -regular cluster, and let R be the constant of Theorem 2.1. Then for each positive $\epsilon \leq Rh^{2d-1}$ the error set $\bar{E}_\epsilon(G) \subset \bar{\mathcal{P}}$ is contained in the moment parallelepiped $\Pi_{\epsilon',h}(G)$, and contains another such parallelepiped $\Pi_{\epsilon'',h}(G)$, where $\epsilon' = (1+|\kappa|)^{2d-1}\epsilon$, $\epsilon'' = (1+|\kappa|)^{-2d+1}\epsilon$.*

Proof: We will describe the effect of a shift by κ on the moment error set ME_ϵ . Define shift transformation $Sh_\kappa : \mathcal{P} \rightarrow \mathcal{P}$ of the parameter space by $Sh_\kappa(F)(x) = F(x - \kappa)$. The following proposition describes the action of the coordinate shift on the moments of general spike-trains (of course, this result remains valid for the moments of any measure on \mathbb{R}).

PROPOSITION 3.1.

$$m_k(F) = \sum_{l=0}^k \binom{k}{l} (-\kappa)^{k-l} m_l(Sh_\kappa(F)), \quad m_k(Sh_\kappa(F)) = \sum_{l=0}^k \binom{k}{l} (\kappa)^{k-l} m_l(F).$$

Proof: For $F(x) = \sum_{j=1}^d a_j \delta(x - x_j) \in \mathcal{P}$ we get

$$m_k(Sh_\kappa(F)) = \sum_{j=1}^d a_j (\kappa + x_j)^k = \sum_{j=1}^d a_j \sum_{l=0}^k \binom{k}{l} \kappa^{k-l} x_j^l =$$

$$= \sum_{l=0}^k \binom{k}{l} \kappa^{k-l} \sum_{j=1}^d a_j x_j^l = \sum_{l=0}^k \binom{k}{l} \kappa^{k-l} m_l(F).$$

Replacing κ by $-\kappa$ we get the second expression. \square

Accordingly, we define the shift transformation $Sh_\kappa^m : \mathcal{M} \rightarrow \mathcal{M}$ of the moment space by the following expression: for $\mu' = (\mu'_0, \dots, \mu'_{2d-1}) \in \mathcal{M}$

$$Sh_\kappa^m(\mu') = \nu = (\nu_0, \dots, \nu_{2d-1}), \quad \nu_k = \sum_{l=0}^k \binom{k}{l} (\kappa)^{k-l} \mu_l, \quad k = 0, 1, \dots, 2d-1.$$

Proposition 3.1 shows that the shift transformations S_κ and S_κ^m and the Prony mapping PM satisfy the following identity:

$$(3.1) \quad PM(Sh_\kappa(F)) = Sh_\kappa^m(PM(F)).$$

We have the following bounds for norm of Sh_κ^m :

PROPOSITION 3.2. *With respect to the maximum norm $|\mu|, |\nu|$ on \mathcal{M} the shift transformation $Sh_\kappa^m : \mathcal{M} \rightarrow \mathcal{M}$ satisfies*

$$|Sh_\kappa^m| \leq (1 + |\kappa|)^{2d-1}.$$

For each $\mu \in \mathcal{M}$, and $\nu = Sh_\kappa^m(\mu)$ we have

$$|\nu| \geq (1 + |\kappa|)^{-2d+1} |\mu|.$$

Proof: We have, for $\mu = (\mu_0, \dots, \mu_{2d-1}) \in \mathcal{M}$, with $|\mu| = \max_{k=0, \dots, 2d-1} |\mu_k| = 1$,

$$\begin{aligned} |Sh_\kappa^m(\mu)| &= \max_{k=0, \dots, 2d-1} |\nu_k| = \max_{k=0, \dots, 2d-1} \left| \sum_{l=0}^k \binom{k}{l} (\kappa)^{k-l} \mu_l \right| \leq \\ &\leq \max_{k=0, \dots, 2d-1} \sum_{l=0}^k \binom{k}{l} |\kappa|^{k-l} = \max_{k=0, \dots, 2d-1} (1 + |\kappa|)^k = (1 + |\kappa|)^{2d-1}. \end{aligned}$$

The second inequality follows via application of the first one to the inverse shift $Sh_{-\kappa}^m$. \square

Put, as above $PM(F) = \mu$, and denote the signal $Sh_\kappa(F)$ by H . Then by (3.1) we have $PM(H) = Sh_\kappa^m(\mu) = \nu$. Now we can describe the set $\bar{ME}_\epsilon(H)$ of the possible measurements for the moments of the shifted signal H , implied by our assumption that the errors in the measured moments of F do not exceed ϵ . Since the set of the possible measurements of F is the ϵ -cube $Q_\epsilon(\mu) \subset \mathcal{M}$, we have $\bar{ME}_\epsilon(H) = Sh_\kappa^m(Q_\epsilon(\mu))$.

LEMMA 3.1. *The set $\bar{ME}_\epsilon(H)$ is contained in the cube $Q_{\epsilon'}(\tilde{\mu})$, and contains the cube $Q_{\epsilon''}(\tilde{\mu})$, where $\epsilon' = (1 + |\kappa|)^{2d-1}\epsilon$, $\epsilon'' = (1 + |\kappa|)^{-2d+1}\epsilon$:*

$$Q_{\epsilon''}(\tilde{\mu}) \subset \bar{ME}_\epsilon(Sh_\kappa(F)) \subset Q_{\epsilon'}(\tilde{\mu}).$$

Proof: It follows directly from Proposition 3.2. \square

Now the cluster of the shifted signal H is centered around zero. The model signal G is obtained from H by an $\frac{1}{h}$ up-scaling. Using Lemma 3.1 and verbally repeating the proof of Theorem 3.1, we complete the proof of Theorem 3.2. \square

3.3. Worst case reconstruction error. As a consequence of Theorem 3.2 we now obtain lower and upper bounds for the worst case reconstruction error $\rho(F, \epsilon)$, defined by

$$\rho(F, \epsilon) = \max_{F' \in E_\epsilon(F)} \|F' - F\|.$$

In a similar way we define $\rho^a(F, \epsilon)$ and $\rho^x(F, \epsilon)$ - the worst case errors in reconstruction of the amplitudes (nodes) of $F = (a, x)$, and $\rho^x(F, \epsilon)$:

$$\rho^a(F, \epsilon) = \max_{F'=(a',x') \in E_\epsilon(F)} \|a' - a\|, \quad \rho^x(F, \epsilon) = \max_{F'=(a',x') \in E_\epsilon(F)} \|x' - x\|.$$

The following theorem provides sharp, up to constants, lower and upper bounds on $\rho(F, \epsilon)$, $\rho^a(F, \epsilon)$, $\rho^x(F, \epsilon)$. For $F \in \mathcal{P}$ forming an (h, κ, η, m, M) -regular cluster let the constants R, C_2, C_3, C_4 be as above, and put $\gamma = \min\{1, \frac{C_5}{10C_4R}\}$,

$$K_1 = \frac{9C_5}{10}(1 + |\kappa|)^{-2d+1}\gamma, \quad K_2 = (1 + |\kappa|)^{2d-1}C_3.$$

Here C_5 is a constant depending only on d, η, m, M . It is defined in the proof of Theorem 3.3, given below.

THEOREM 3.3. *Let $F \in \mathcal{P}$ form an (h, κ, η, m, M) -regular cluster. Then for each positive $\epsilon \leq Rh^{2d-1}$ the following bounds for the worst case reconstruction errors are valid:*

$$K_1\epsilon h^{-2d+1} \leq \rho(F, \epsilon), \rho^a(F, \epsilon) \leq K_2\epsilon h^{-2d+1},$$

$$K_1\epsilon h^{-2d+2} \leq \rho^x(F, \epsilon) \leq K_2\epsilon h^{-2d+2}.$$

In particular, for $\epsilon = Rh^{2d-1}$ we get

$$K_1R \leq \rho(F, \epsilon), \rho^a(F, \epsilon) \leq K_2R, \quad K_1Rh \leq \rho^x(F, \epsilon) \leq K_2Rh.$$

Proof: Let G be the model signal of F , obtained by κ -shift and $\frac{1}{h}$ up-scaling of F . We define the (model) worst case reconstruction error $\bar{\rho}(G, \epsilon)$ by

$$\bar{\rho}(G, \epsilon) = \max_{G' \in \bar{E}_\epsilon(G)} \|G' - G\|,$$

and the (model) worst case reconstruction error $\tilde{\rho}(G, \epsilon)$ in the moment metric by

$$\tilde{\rho}(G, \epsilon) = \max_{G' \in \bar{E}_\epsilon(G)} d(G', G).$$

Via comparison of moment and Euclidean metrics, we conclude that

$$(3.2) \quad \bar{\rho}(G, \epsilon) \leq C_3\tilde{\rho}(G, \epsilon).$$

By Theorem 3.2, the error set $\bar{E}_\epsilon(G)$ contains the moment parallelepiped $\Pi_{\epsilon'', h}(G)$, and is contained in $\Pi_{\epsilon', h}(G)$, with $\epsilon' = (1 + |\kappa|)^{2d-1}\epsilon$, $\epsilon'' = (1 + |\kappa|)^{-2d+1}\epsilon$. Therefore we have

$$\tilde{\rho}(G, \epsilon) \leq \max_{G' \in \Pi_{\epsilon', h}(G)} d(G', G) = \epsilon'' h^{-2d+1},$$

and, via (3.2),

$$\bar{\rho}(G, \epsilon) \leq C_3\epsilon' h^{-2d+1} = K_2\epsilon h^{-2d+1}.$$

Now we return from G to the original signal F , and from the space $\bar{\mathcal{P}}$ to \mathcal{P} . In this transformation the amplitudes a remain unchanged, while the nodes x are multiplied by $h \leq 1$. Hence the Euclidean norm of vectors (a, x) may only decrease in this transformation. Therefore the previous inequality implies

$$(3.3) \quad \rho(F, \epsilon) \leq K_2\epsilon h^{-2d+1}.$$

This proves the upper bound of the first inequality of Theorem 3.3. To complete the proof we have to investigate separately the behavior of the amplitudes a and the nodes x .

Put $\epsilon''' = \gamma\epsilon''$. We now fix G' to be the endpoint of the Prony curve $S_{2d-2}(G)$ inside the moment parallelepiped $\Pi_{\epsilon''',h}(G)$. $S_{2d-2}(G)$ is defined by the equations $m_k(G') = m_k(G) = \nu_k$, $k = 0, \dots, 2d-2$, so the moment coordinates of G' are $\nu_0, \dots, \nu_{2d-2}, \nu_{2d-2} + \epsilon'''h^{-2d+1}$. Since the moment parallelepiped $\Pi_{\epsilon'',h}(G)$ is contained in the error set $\bar{E}_\epsilon(G)$, and by definition $\gamma \leq 1$, we have $\Pi_{\epsilon''',h}(G) \subset \Pi_{\epsilon'',h}(G) \subset \bar{E}_\epsilon(G)$, and hence G' belongs to the error set $\bar{E}_\epsilon(G)$.

Writing $G = (\bar{a}, \bar{x})$, $G' = (a', x')$ we now prove the following lower bound:

LEMMA 3.2. *The amplitudes and nodes of G' satisfy*

$$(3.4) \quad K_1\epsilon h^{-2d+1} \leq \|a' - \bar{a}\|, \quad \|x' - \bar{x}\| \leq K_2\epsilon h^{-2d+1}.$$

Proof: The upper bound follows immediately from (3.3), since $(\bar{a}, \bar{x}), (a', x')$ are the projections of G, G' to the amplitudes (nodes) subspace $\bar{\mathcal{P}}^a$ ($\bar{\mathcal{P}}^x$, respectively) of $\bar{\mathcal{P}}$. In order to obtain the lower bound we study in more detail the form of the Jacobian J of the Prony mapping, and then use a comparison between J^{-1} and PM^{-1} provided by Theorem 2.1.

The Jacobian J of PM at the point $G = (\bar{a}, \bar{x})$ is given by the matrix MJ . The upper left $d \times d$ minor MJ_1 of MJ is nonzero, being the Vandermonde matrix on the nodes $\bar{x}_1, \dots, \bar{x}_d$. It can be explicitly bounded from zero in terms of d, η, m, M . The same is true for MJ_2 - the upper right minor, shifted one step down: after factoring out the common coefficients of the rows and the columns we get once more the Vandermonde matrix on the nodes $\bar{x}_1, \dots, \bar{x}_d$.

Consider now the point $v = \nu + (0, \dots, 0, \epsilon'''h^{-2d+1}) \in \mathcal{M}$. We have $G' = PM^{-1}(v) = (a', x')$. Let $w = J^{-1}(v) = (\hat{a}, \hat{x})$. An easy estimate using the minors MJ_1 and MJ_2 provides a constant C_5 , depending only on d, η, m, M , such that

$$(3.5) \quad \|\hat{a} - \bar{a}\|, \quad \|\hat{x} - \bar{x}\| \geq C_5\epsilon'''h^{-2d+1}.$$

Now we use a comparison between J^{-1} and PM^{-1} provided by Theorem 2.1: for each $\nu' \in Q_R(G)$ we have

$$\|PM^{-1}(\nu') - J^{-1}(\nu')\| \leq C_4\|\nu' - \nu\|^2.$$

Applied to the point $v \in \mathcal{M}$, this bound gives

$$(3.6) \quad \|a' - \hat{a}\|, \quad \|x' - \hat{x}\| \leq \frac{C_5}{10}\epsilon'''h^{-2d+1}.$$

Indeed, since projections do not increase the Euclidean norm, we have

$$\begin{aligned} \|a' - \hat{a}\|, \quad \|x' - \hat{x}\| &\leq \|PM^{-1}(v) - J^{-1}(v)\| \leq C_4(\epsilon'''h^{-2d+1})^2 \leq \\ &\leq C_4\gamma(\epsilon'''h^{-2d+1})(\epsilon''h^{-2d+1}) \leq C_4R\gamma(\epsilon'''h^{-2d+1}) \leq \frac{C_5}{10}\epsilon'''h^{-2d+1}, \end{aligned}$$

since by our assumptions $\epsilon''h^{-2d+1} \leq \epsilon h^{-2d+1} \leq R$, while $\gamma = \min\{1, \frac{C_5}{10C_4R}\}$.

Combining (3.5) and (3.6) we have

$$\|a' - \bar{a}\| \geq \|\hat{a} - \bar{a}\| - \|a' - \hat{a}\| \geq C_5\epsilon'''h^{-2d+1} - \frac{C_5}{10}\epsilon'''h^{-2d+1} = \frac{9C_5}{10}\epsilon'''h^{-2d+1}.$$

Exactly in the same way we get also $\|x' - \bar{x}\| \geq \frac{9C_5}{10}\epsilon'''h^{-2d+1}$. We can rewrite the right hand side as

$$\frac{9C_5}{10}\epsilon'''h^{-2d+1} = \frac{9C_5}{10}\gamma(1 + |\kappa|)^{-2d+1}\epsilon h^{-2d+1}.$$

So putting $K_1 = \frac{9C_5}{10}(1 + |\kappa|)^{-2d+1}\min\{1, \frac{C_5}{10C_4R}\}$ we obtain

$$\|a' - \bar{a}\|, \|x' - \bar{x}\| \geq K_1\epsilon h^{-2d+1}.$$

This completes the proof of Lemma 3.2. \square

It remains to return from G to the original signal F , and from the space $\bar{\mathcal{P}}$ to \mathcal{P} . In this transformation the amplitudes a remain unchanged, while the nodes x are multiplied by h . Hence the bound of Lemma 3.2 becomes

$$K_1\epsilon h^{-2d+1} \leq \|a' - \bar{a}\| \leq K_2\epsilon h^{-2d+1}, \quad K_1\epsilon h^{-2d+2} \leq \|x' - \bar{x}\| \leq K_2\epsilon h^{-2d+2}.$$

This completes the proof of Theorem 3.3. \square

Till now we have assumed that all the d nodes of the signal F form a cluster of size h . The *lower bounds* of Theorem 3.3 can be easily extended to the case where there are also non-cluster nodes:

COROLLARY 3.1. *Let $F \in \mathcal{P}_d$. Assume that some $s \leq d$ of the nodes of F form an (h, κ, η, m, M) -regular cluster. Then for each positive $\epsilon \leq Rh^{2d-1}$ we have*

$$\rho(F, \epsilon), \rho^a(F, \epsilon) \geq K_1\epsilon h^{-2s+1}, \quad \rho^x(F, \epsilon) \geq K_1\epsilon h^{-2s+2},$$

with the same constants R, K_1 , depending on κ, η, m, M , as above.

Proof: The required lower bound follows directly from Theorem 3.3. Indeed, we can perturb only the nodes and the amplitudes in the cluster, leaving the other nodes and amplitudes fixed, and then all the calculations and estimates above remain unchanged. \square

Remark In the presence of non-cluster nodes obtaining the *upper bounds* for the worst case reconstruction error requires additional considerations. Indeed, perturbing both the cluster and the non-cluster nodes and the amplitudes a priori may create even larger deviations than those of Theorem 3.3, with the moments, remaining within ϵ of the original ones. Accuracy estimates in this situation presumably require analysis of several geometric scales at once. There are important open questions related to this multi-scale analysis. In particular, the following question was suggested in [11]: is it true (as numerical experiments suggest) that for well-separated non-cluster nodes, the accuracy of their reconstruction in Prony inversion is of order ϵ , independently of the size and structure of the cluster?

Our next result concerns the worst case accuracy of reconstruction of the Prony leaves $S_q(F)$. The point is that the smaller is q the larger is the variety $S_q(F)$, but the higher is the accuracy of its reconstruction. In the next section we show how this fact can help in improving the reconstruction accuracy of the signal F itself.

We will state this result only in the normalized signal space $\bar{\mathcal{P}}$. We define the worst case error $\rho_q(G, \epsilon)$ in reconstruction of the Prony leave $S_q(G)$ by

$$\rho_q(G, \epsilon) = \max_{G' \in \bar{E}_\epsilon(G)} d_H(S_q(G'), S_q(G)),$$

where d_H is the Hausdorff distance between the two sets.

THEOREM 3.4. *Let $F \in \mathcal{P}$ form an (h, κ, η, m, M) -regular cluster. Then for each positive $\epsilon \leq Rh^{2d-1}$*

$$K_1 \epsilon h^{-q} \leq \rho_q(\hat{F}, \epsilon) \leq K_2 \epsilon h^{-q},$$

with the same R, K_1, K_2 as above.

Proof: The Prony leaves $S_q(G'), S_q(G) \subset \bar{\mathcal{P}}$ are the moment coordinate subspaces given by $m_k = \bar{m}_k$, $k = 0, \dots, q$, ($m_k = m'_k$, $k = 0, \dots, q$, respectively). The Hausdorff distance between them, with respect to the moment metric d , is equal to the $\max_{k=0, \dots, q} |m'_k - \bar{m}_k| \leq \epsilon h^{-q}$. Now using Theorem 3.2, and verbally repeating the proof of Theorem 3.3, we obtain the required bounds. \square

Notice that, essentially, Theorem 3.3 is a special case of Theorem 3.4, for $q = 2d - 1$, besides the separate bounds for the amplitudes and the nodes in the first theorem, which we do not address in the second one.

3.4. How to use additional information. In this section we shortly describe the way one can use the Prony leaves in order to improve the reconstruction accuracy. Two specific examples are given in the next section.

Assume that we have a certain additional information on the signals F to be reconstructed. We do not specify here the nature of this information, assuming just that a certain functional $\omega(F')$ is known to be positive on the reconstructed signals.

We can analyze in advance the behavior of the functional ω on the Prony leaves S_q of different orders $q = 0, 1, \dots, 2d - 2$, and then we chose and fix a specific value of q , for which we expect the most significant improvement.

Assume now that we are given noisy measurements $\mu' = (\mu'_0, \dots, \mu'_{2d-1})$ of a signal F , with $PM(F) = \mu = (\mu_0, \dots, \mu_{2d-1})$, and let $F' = PM^{-1}(\mu') \in E_\epsilon(F)$ be the Prony reconstruction of μ' . We know that the true solution F belongs to the error set $E_\epsilon(F')$.

As the first step, we construct the Prony leave $S_q(F')$ containing F' .

Next we consider the intersections $E_{\epsilon, q}(F')$ of the leave $S_q(F')$ with the error set $E_\epsilon(F')$, and inside it the “feasible error set”

$$E_{\epsilon, q}^\omega(F') = \{F'' \in E_{\epsilon, q}(F'), \omega(F'') > 0\}.$$

Finally, if $F' \in E_{\epsilon, q}^\omega(F')$, it remains to be the output reconstruction. Otherwise we find the point $F'' \in E_{\epsilon, q}^\omega(F')$ which is the closest to F' , and replace F' by F'' .

We know from Theorem 3.4 that the reconstruction accuracy of the leave S_q is of order ϵh^{-q} , which is h^{2d-1-q} times better than the worst case error of solving the full Prony system. In other words, the distance between $S_q(F')$ and the (unknown) true leave $S_q(F)$ is of order ϵh^{-q} . The output signal F'' belongs to $S_q(F')$, and hence the accuracy of reconstruction of the first q moment coordinates (which is of order ϵh^{-q}) is preserved. The accuracy with respect to the remaining $2d - 1 - q$ moment coordinates (“along S_q ”) may be improved, depending on the size of the feasible error set $E_{\epsilon, q}^\omega(F')$ on $S_q(F')$.

In the next section we illustrate this general approach with specific examples in case $d = 2, q = 2$ and ω being the maximal (or minimal) size of the amplitudes.

4. Prony curves and error amplification in case of two nodes

We consider signals

$$F(x) = a_1\delta(x - x_1) + a_2\delta(x - x_2) \in \mathcal{P}_2.$$

Put $m_k = m_k(F)$, $k = 0, 1, 2$, then the Prony curve $S_2(F)$ is defined by the system of equations

$$(4.1) \quad \begin{aligned} a_1 + a_2 &= m_0 \\ a_1x_1 + a_2x_2 &= m_1 \\ a_1x_1^2 + a_2x_2^2 &= m_2 \end{aligned}$$

Algebraic curves $S_2(F)$ in \mathcal{P}_2 , which we will denote also $S(m_0, m_1, m_2)$, or shortly by S , are defined for any moments m_0, m_1, m_2 . They can be studied globally, using explicit algebraic parametrization. Since we are interested in the behavior of the nodes x_1, x_2 along the curves S , we will consider also the node parameter space $\mathcal{P}_2^x = \{(x_1, x_2)\}$, and the projections $S^x \subset \mathcal{P}_2^x$ of the Prony curves S to the node space \mathcal{P}_2^x . The following result was proved in [3]:

PROPOSITION 4.1. *The curves $S^x(m_0, m_1, m_2) \subset \mathcal{P}_2^x$ are hyperbolas in the plane x_1, x_2 defined by the equation*

$$(4.2) \quad m_0x_1x_2 - m_1(x_1 + x_2) + m_2 = 0.$$

The corresponding curves $S(m_0, m_1, m_2) \subset \mathcal{P}_2$ are parametrized as

$$(4.3) \quad a_1 = \frac{m_0x_2 - m_1}{x_2 - x_1}, \quad a_2 = \frac{-m_0x_1 + m_1}{x_2 - x_1}, \quad (x_1, x_2) \in S^x(m_0, m_1, m_2).$$

The explicit description of the Prony leaves S_q given in Proposition 4.1, can be extended to the case of more than two nodes. It also can be combined with the analysis of the Prony mapping from the point of view of Singularity Theory, given in [8, 34], including, in particular, representation of signals F in the “bases of finite differences” introduced in [8, 34]. We plan to present these results separately.

In this paper we just give a few typical examples of a behavior of the Prony reconstruction on and near the curves S_2 . We show that understanding this behavior may help to improve the reconstruction accuracy, if some a priori information is available.

4.1. Escape of one node to infinity. Consider $F(x) = \frac{1}{2}\delta(x+h) + \frac{1}{2}\delta(x-h)$. Then $m_0(F) = 1, m_1(F) = 0, m_2(F) = h^2$. By Proposition 4.1 the curve $S^x = S^x(1, 0, h^2)$ is a hyperbola $x_1x_2 = -h^2$ in the plane x_1, x_2 , while the parametrization of the curve $S = S(1, 0, h^2) \subset \mathcal{P}_2$ is given by

$$a_1 = \frac{x_2}{x_2 - x_1}, \quad a_2 = \frac{-x_1}{x_2 - x_1}, \quad (x_1, x_2) \in S^x.$$

We put $x_1 = uh$, with u a parameter tending to zero (for $u = -1$ we get the original signal F). On the hyperbola S^x we have $x_2 = -\frac{h}{u}$, so the second node escapes to infinity, as $u \rightarrow 0$. We are interested in the behavior of the signal

$F_u(x)$, corresponding to the point $(a_1, a_2, uh, -\frac{h}{u})$ on the curve S , as $u \rightarrow 0$. By Proposition 4.1 we get

$$a_1 = \frac{-\frac{h}{u}}{-\frac{h}{u} - uh} \rightarrow 1, \quad a_2 = -\frac{uh}{-\frac{h}{u} - uh} \sim u^2, \quad F_u(x) \sim \delta(x - hu) + u^2 \delta(x + \frac{h}{u}).$$

The first three moments m_0, m_1, m_2 are constant along the curve S by definition. Hence the error occurs only in the fourth moment $m_3(F_u)$. We get

$$m_3(F_u) = a_1 x_1^3 + a_2 x_2^3 \sim h^3 u^3 - u^2 \frac{h^3}{u^3} \sim -\frac{h^3}{u}$$

for small u . In particular, for $u \sim -h$ we see that the first node $x_1 \sim -h^2$, the second node $x_2 \sim 1$, while $m_3(F_u) \sim h^2$. Notice that $m_3(F) = h^3$, and hence the error, which is the difference $|m_3(F_u) - m_3(F)| \sim h^2$.

The behavior of the signal $F_u(x)$ as $u \rightarrow -\infty$, i.e. $x_1 \rightarrow -\infty$, $x_2 \rightarrow 0$, is completely symmetric.

We conclude that for the measurements error $\epsilon \sim h^2$ one of the nodes may jump to the distance $\sim h$ of its true position, to the central point between the two true nodes. Its new amplitude becomes the sum of the true amplitudes. The second node escapes to the distance ~ 1 from the true position, while its amplitude decreases to $\sim h^2$.

This type of behavior appears in numerical inversion of Prony systems.

Assume now that we have a certain additional information on the signal F to be reconstructed. Then, using the description of the geometry of the error amplification given above, it may be possible to achieve a serious improvement in reconstruction accuracy. As an illustrative example, assume that we know a priori that the absolute values of the amplitudes $|a_1|, |a_2|$ cannot be smaller than $\sigma > 0$. Since we have $a_2 \sim u^2$, we conclude that $|u| \geq \sqrt{\sigma}$, and hence $|x_2| = \frac{h}{|u|} \leq \frac{h}{\sqrt{\sigma}}$. This condition significantly restricts the possible position of the reconstructed signals.

A simple algorithm in the lines of Section 3.4, utilizing the restriction above, is the following:

Step 1. For given noisy measurements $\mu' = (\mu'_0, \mu'_1, \mu'_2, \mu'_3)$ construct, via Proposition 4.1, the Prony curve S' .

Step 2. Find the signal $F' = PM^{-1}(\mu')$ solving the Prony system (by solving on the Prony curve S' the fourth Prony equation).

Step 3. If F' satisfies the condition $|a'_1|, |a'_2| \geq \sigma$, then the output of the algorithm is F' .

Step 3. If F' violates the condition $|a'_1|, |a'_2| \geq \sigma$, the output of the algorithm is the signal F'' on the curve S' , which satisfies $|a''_1|, |a''_2| \geq \sigma$, and is the closest to F' .

4.2. Collision of the nodes. Consider $F(x) = \frac{4}{3}\delta(x - \frac{1}{2}h) - \frac{1}{3}\delta(x - 2h)$. Then $m_0(F) = 1, m_1(F) = 0, m_2(F) = -h^2$. The curve $S^x = S^x(1, 0, -h^2)$ is a hyperbola $x_1 x_2 = h^2$ in the plane x_1, x_2

In contrast to the case considered in Section 4.1, now the hyperbola S^x crosses the diagonal $\Delta = \{x_1 = x_2\}$ at the point (h, h) . Thus along this curve we encounter the collision of the nodes. We put $x_1 = h(1+u)$, with u a parameter tending to zero.

In this case on the hyperbola S^x we have $x_2 = \frac{h}{1+u} \sim h(1-u)$, and $x_2 - x_1 \sim -2u$, tending to 0 as $u \rightarrow 0$. For the amplitudes we have

$$a_1 \sim \frac{u-1}{2u} \rightarrow \infty, a_2 \sim \frac{u+1}{2u} \rightarrow \infty, F_u(x) \sim \frac{u-1}{2u} \delta(x-h-hu) + \frac{u+1}{2u} \delta(x-h+hu).$$

For the third moment of F_u we get

$$m_3(F_u) = a_1 x_1^3 + a_2 x_2^3 \sim \frac{1}{2} h^3 \left(\frac{u-1}{u} (1+u)^3 + \frac{u+1}{u} (1-u)^3 \right) \rightarrow -4h^3, u \rightarrow 0.$$

In particular, for small u we have $m_3(F_u) \sim -4h^3$. Since $m_3(F) = -\frac{23}{24}h^3$, the difference $|m_3(F_u) - m_3(F)| \sim h^3$.

We conclude that already for the measurements error $\epsilon \sim h^3$ the two nodes x_1, x_2 may collide. The amplitudes a_1, a_2 in this case tend to $\pm\infty$ as one over twice the distance $|x_2 - x_1|$.

This type of behavior occurs in numerical inversion of Prony systems.

As an illustrative example, assume that we know a priori that the absolute values of the amplitudes $|a_1|, |a_2|$ cannot be bigger than $M > 0$. Since we have $|a_1|, |a_2| \sim \frac{1}{u}$, we conclude that $|u| \geq \frac{1}{M}$, and hence the distance of x_1 to the diagonal Δ is at least $\frac{1}{M}h$. This condition significantly restricts the possible position of the reconstructed signals.

A simple algorithm, utilizing this restriction, is completely similar to the algorithm described in Section 4.1 above.

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